Wronskian formula for confluent second-order supersymmetric quantum mechanics

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Abstract

The confluent second-order supersymmetric quantum mechanics, with factorization energies $\epsilon_1,\ \epsilon_2$ tending to a single ϵ-value, is studied. We show that the Wronskian formula remains valid if generalized eigenfunctions are taken as seed solutions. The confluent algorithm is used to generate SUSY partners of the Coulomb potential.

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1 Introduction.

The supersymmetric quantum mechanics (SUSY QM) represents a powerful tool for the spectral design in quantum theory. It has proved successful to construct solvable potentials for which the spectral information can be anallytically determined [1–5]. The simplest version is the first-order SUSY QM which, however, does not allow to modify levels different from the ground state without creating singularities in the SUSY partner potential [3]. As an alternative to surpass this problem, the second-order SUSY QM can be used [6–13]. Indeed, through this technique one can embed two levels $\epsilon_1,\ \epsilon_2$ between two neighbouring energies of the initial Hamiltonian [5, 7]. It is possible as well to create single levels above E_0 [13] and to generate complex potentials with either purely real spectra or having some complex 'energies' [12].

For $\epsilon_1 \neq \epsilon_2$ the modification to the potential induced by the second-order SUSY QM involves the Wronskian of the associated seed Schrödinger solutions $u_1,\ u_2$. In the confluent case $\epsilon_1 = \epsilon_2 \equiv \epsilon$, however, the key function $w(x)$ inducing the change depends just of one seed solution u and it is not clear that the treatment based on the Wronskian remains valid. In this paper we will prove that result by identifying appropriate (generalized) eigenfunctions

for which the Wronskian coincides with $w(x)$. This implies that a unified technique valid in the non-confluent and in the confluent case is available, we just have to identify the right seed solutions.

In the next section we will address the second-order SUSY QM and its classification scheme. We will study in some detail the confluent case, the restrictions onto the seed solution to ensure the regularity of the potentials difference and the discussion about the Wronskian. Then, we will generate confluent second-order SUSY partners of the Coulomb potential (some of which are new). We will end the paper with our conclusions.

2 Second-order supersymmetric quantum mechanics.

The second-order SUSY QM consists in the following realization of the standard SUSY algebra with two generators:

$$\{Q_j, Q_k\} = \delta_{jk} H_{ss}, \quad [H_{ss}, Q_j] = 0, \quad j,k = 1,2 \tag{1}$$

$$Q_1 = \frac{Q^\dagger + Q}{\sqrt{2}}, \quad Q_2 = \frac{Q^\dagger - Q}{i\sqrt{2}}, \quad Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix} \tag{2}$$

$$H_{ss} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \prod_{j=1}^{2} \begin{pmatrix} \widetilde{H} - \epsilon_j & 0 \\ 0 & H - \epsilon_j \end{pmatrix} \tag{3}$$

$$H = -\frac{d^2}{dx^2} + V(x), \quad \widetilde{H} = -\frac{d^2}{dx^2} + \widetilde{V}(x) \tag{4}$$

$$A = \frac{d^2}{dx^2} + \eta(x)\frac{d}{dx} + \gamma(x), \quad \widetilde{H}A = AH \tag{5}$$

The intertwining relationship (5) leads to the following set of equations:

$$\frac{\eta\eta''}{2} - \frac{\eta'^2}{4} + \eta^2\left(\frac{\eta^2}{4} - \eta' - V + d\right) + c = 0 \tag{6}$$

$$\gamma = d - V + \frac{\eta^2}{2} - \frac{\eta'}{2}, \qquad \widetilde{V} = V + 2\eta' \tag{7}$$

Given $V(x)$, c and d, the new potential $\widetilde{V}(x)$ is determined by the solutions η to the non-linear equation (6). To find them, let us use the Ansätz:

$$\eta'(x) = \eta^2(x) + 2\beta(x)\eta(x) - 2\xi(x) \tag{8}$$

which leads to $\xi^2 = c$ and the following Riccati equation:

$$\beta' + \beta^2 = V - \epsilon, \quad \epsilon = d + \xi \tag{9}$$

We can work instead with the equivalent Schrödinger equation ($\beta = u'/u$)

$$-u'' + Vu = \epsilon u \tag{10}$$

Thus, the solutions η can be classified according to the sign of c.

3 Classification of second-order SUSY transformations.

3.1 The non-confluent cases with $c \neq 0$.

Here $\epsilon_1 \equiv d+\sqrt{c},\ \epsilon_2 \equiv d-\sqrt{c},\ \epsilon_1 \neq \epsilon_2$; this includes the real case with $c>0,\ \epsilon_1, \epsilon_2 \in \mathbb{R}$ and the complex one with $c<0,\ \epsilon_1 \in \mathbb{C}, \epsilon_2 = \epsilon_1^*$. From (8) two equations for η are obtained:

$$\eta' = \eta^2 + 2\beta_1\eta - (\epsilon_1 - \epsilon_2), \qquad \eta' = \eta^2 + 2\beta_2\eta + (\epsilon_1 - \epsilon_2)$$

which leads to:

$$\eta = \frac{\epsilon_1 - \epsilon_2}{\beta_1 - \beta_2} = -\frac{d}{dx}\ln[W(u_1, u_2)] \tag{11}$$

where $W(f,g) = fg' - gf'$ is the Wronskian of f and g. To avoid singularities in $\eta(x)$ the Wronskian has to be nodeless. The spectrum of $\widetilde{H}$, $\mathrm{Sp}(\widetilde{H})$, will depend as well on the normalizability of the two eigenfunctions $\widetilde{\psi}_{\epsilon_j}$ of $\widetilde{H}$ with eigenvalue ϵ_j in the Kernel of $A^\dagger$, with explicit expressions given by $\widetilde{\psi}_{\epsilon_1} \propto u_2/W(u_1,u_2),\ \widetilde{\psi}_{\epsilon_2} \propto u_1/W(u_1,u_2)$.

Some spectral design possibilities are worth to be mentioned (for a detailed treatment see [5]). *i)* Two levels $\epsilon_1,\ \epsilon_2$ can be created *below* the ground state energy of H, namely, $\epsilon_1 < \epsilon_2 < E_0$, $\mathrm{Sp}(\widetilde{H}) = \{\epsilon_1, \epsilon_2, E_n, n = 0,1,2,\cdots\}$ [8]. *ii)* A pair of levels can be placed *between* two neighbouring energies of H, i.e., $E_i < \epsilon_1 < \epsilon_2 < E_{i+1}$, $\mathrm{Sp}(\widetilde{H}) = \{E_0, \cdots, E_i, \epsilon_1, \epsilon_2, E_{i+1}, \cdots\}$ [7]. *iii)* Two neighbouring energies of H can be *deleted*, namely, $\epsilon_1 = E_i,\ \epsilon_2 = E_{i+1}$, $\mathrm{Sp}(\widetilde{H}) = \{E_0, \cdots, E_{i-1}, E_{i+2}, \cdots\}$ [5]. *iv)* Some *complex* energies can be manufactured (the new Hamiltonians are non-hermitian) [12].

Cases *ii), iii), iv)* run against the dominant idea that in SUSY QM the new levels are always real and *below* the ground state energy of the initial Hamiltonian, improving thus our spectral design possibilities [4,5].

3.2 The confluent case with $c = 0$.

We have now that $\epsilon \equiv \epsilon_1 = \epsilon_2$, and the Ansätz (8) just provides [10,13]:

$$\eta' = \eta^2 + 2\beta\eta$$

This Bernoulli equation has a general solution given by

$$\eta(x) = \frac{e^{2\int\beta(x)dx}}{w_0 - \int e^{2\int\beta(x)dx}dx} = -\frac{d}{dx}\ln[w(x)] \tag{12}$$

where, up to an unimportant constant factor,

$$w(x) = w_0 - \int_0^x u^2(y)dy \tag{13}$$

Since equations (11,12) look similar, perhaps $w(x)$ has to do with a Wronskian. To see that, suppose there is a function v related with the given u through:

$$(H-\epsilon)v = u, \quad (H-\epsilon)u = 0 \tag{14}$$

i.e., u and v are rank 1 and rank 2 generalized eigenfunctions of H. This fact was noticed in [4], but a further study exploring the link with the Wronskian formula is needed. Let us find now v from the differential equation (14):

$$v = u\left(k + \int \frac{w(x)}{u^2(x)}dx\right) \tag{15}$$

Up to a constant factor, it turns out that $w(x) = W(u, v)$.

Similarly as for $\epsilon_1 \neq \epsilon_2$, the regular confluent transformation will be produced by a nodeless $w(x)$. If the domain of x is $\mathbb{R}$, it is sufficient to use solutions $u(x)$ vanishing either when $x \to -\infty$ or when $x \to \infty$ [13]. However, if the x-domain is the positive semi-axis, as in the Coulomb problem which we will address later, the simplest choice is to take:

$$\lim_{x\to 0} u(x) = 0 \quad \text{or} \quad \lim_{x\to\infty} u(x) = 0 \tag{16}$$

For $\epsilon \notin \text{Sp}(H)$ one of the requirements (16) can be achieved, but not both simultaneously. For $\epsilon \in \text{Sp}(H)$ the two conditons (16) are automatically satisfied. In both cases a normalized eigenfunction of $\widetilde{H}$ with eigenvalue ϵ in the Kernel of $A^\dagger$ can be found, $\widetilde{\psi}_\epsilon \propto u/w$. In particular, for $\epsilon \geq E_0$ equation (16) can be fulfilled, i.e., one level can be created above E_0. Let us remind that the excited state solutions obey (16) and thus they are appropriate to generate new potentials through the confluent SUSY algorithm.

4 Confluent SUSY partners of the Coulomb potential.

Let us apply the confluent algorithm to the Coulomb problem which, after separating the angular variables and taking $\hbar = e = m = 1$, leads to a Schrödinger Hamiltonian as given in (4), with an effective potential

$$V(r) = -\frac{2}{r} + \frac{\ell(\ell+1)}{r^2} \tag{17}$$

where $\ell = 0, 1, \ldots$. The spectrum of the corresponding Hamiltonian reads $\text{Sp}(H) = \{E_n = -1/n^2,\ n = \ell+1, \ell+2, \ldots\}$. First we find the general solution to (10) with the $V(r)$ of (17) and an arbitrary factorization energy $\epsilon < 0$; second, the solution with the right behaviour is chosen (see Eq. (16)). After implementing the procedure, the solution vanishing at the origin reads:

$$u(r) = \sqrt{\frac{(-\epsilon)\Gamma\left(\ell+1+\frac{1}{\sqrt{-\epsilon}}\right)}{\Gamma\left(\frac{1}{\sqrt{-\epsilon}}-\ell\right)[\Gamma(2\ell+2)]^2}}\left(2r\sqrt{-\epsilon}\right)^{\ell+1} e^{-r\sqrt{-\epsilon}}\ {}_1F_1\left(\ell+1-\tfrac{1}{\sqrt{-\epsilon}}, 2\ell+2; 2r\sqrt{-\epsilon}\right) \tag{18}$$

$\Gamma(z)$ and ${}_1F_1(a, b; z)$ being the Gamma and the confluent hypergeometric functions respectively. Notice that $u(r)$ becomes the normalized eigenfunction of H for $\epsilon = E_n$. A straightforward calculation leads now to our key w-function:

$$\begin{aligned} w(r) &= w_0 - \sum_{m=0}^{\infty} \frac{\sqrt{-\epsilon}\, B\left(\ell+1+\frac{1}{\sqrt{-\epsilon}}, \ell+1+m-\frac{1}{\sqrt{-\epsilon}}\right)(2r\sqrt{-\epsilon})^{2\ell+m+3}}{2(2\ell+m+3)(2\ell+1)!\, m!\, B\left(\frac{1}{\sqrt{-\epsilon}}-\ell, \ell+1-\frac{1}{\sqrt{-\epsilon}}\right)} \\ &\quad \times {}_2F_2\left(2\ell+m+3, \ell+1+\frac{1}{\sqrt{-\epsilon}}; 2\ell+m+4, 2\ell+2; -2r\sqrt{-\epsilon}\right) \end{aligned} \tag{19}$$

where $B(x,y)$, ${}_2F_2(a_1,a_2;b_1,b_2;z)$ are the Beta and a generalized hypergeometric function. For $\epsilon = E_n = -1/n^2$ this infinite series truncates:

$$
\begin{aligned}
w(r) &= w_0 - \sum_{m=0}^{n-\ell-1} \frac{(-1)^m \left(\frac{2r}{n}\right)^{2\ell+m+3}}{(2\ell+m+3)(2\ell+m+1)} \\
&\qquad \times \frac{{}_2F_2(2\ell+m+3, n+\ell+1; 2\ell+m+4, 2\ell+2; -\frac{2r}{n})}{2n(2\ell+1)!m!B(n-\ell-m, 2\ell+m+1)}
\end{aligned}
\tag{20}
$$

The w_0-domain where $w(r)$ is nodeless reads $(-\infty, 0]$ for $\epsilon \neq E_n$ and $(-\infty,0] \cup [1,\infty)$ for $\epsilon = E_n$.

The SUSY partner potentials take the form:

$$
\widetilde{V}(r) = -\frac{2}{r} + \frac{\ell(\ell+1)}{r^2} + \frac{4u(r)u'(r)}{w(r)} + \frac{2u^4(r)}{w^2(r)} \tag{21}
$$

In (21) $u(r)$, $w(r)$ are given by (18) and (19,20) while $u'(r)$ arises from (18). Some examples, induced by the $w(r)$ of (20), deserve an explicit discussion.

4.1 The case with $\ell = 0$ and $n = 1$.

Here the SUSY partner potentials of $V(r) = -2/r$ are given by:

$$
\widetilde{V}(r) = -\frac{2}{r} - \frac{16\, r\, [-1 - r + (w_0-1)(r-1)e^{2r}]}{[1 + 2r + 2r^2 + (w_0-1)e^{2r}]^2} \tag{22}
$$

The potentials $V(r)$, $\widetilde{V}(r)$ are isospectral for $w_0 \in (-\infty, 0) \cup (1, \infty)$ and they differ in the ground state for $w_0 = 0, 1$ because $E_1 = -1$ is missing of $\widetilde{H}$ in the last case. The potentials (22) coincide with a family derived long ago through the factorization method, we just take $w_0 = 4\gamma_1$ in (22) to get equation (3.1) of [14] (see also [12, 15, 16]).

4.2 The case with $\ell = 0$ and $n = 2$.

The SUSY partner potentials of $V(r) = -2/r$ become now

$$
\widetilde{V}(r) = -\frac{2}{r} + \frac{8r(r-2)\,[-8 + 4r + 6r^2 + 2r^3 + r^4 - 2(w_0-1)\,(4 - 6r + r^2)\,e^r]}{[8 + 8r + 4r^2 + r^4 + 8(w_0-1)e^r]^2} \tag{23}
$$

Once again, $V(r)$ and $\widetilde{V}(r)$ are isospectral for $w_0 \in (-\infty, 0) \cup (1, \infty)$, and for $w_0 = 0, 1$ their spectra differ because the level $E_2 = -1/4$ is missing of $\widetilde{H}$. As far as we know, the potentials (23) have not been derived previously.

Other families of exactly solvable potentials can be written explicitly for different values of n and ℓ. We illustrate one of them in figure 1 for $n = 4$, $\ell = 1$ and $w_0 = -0.1$. For comparison, the initial potential is also shown.

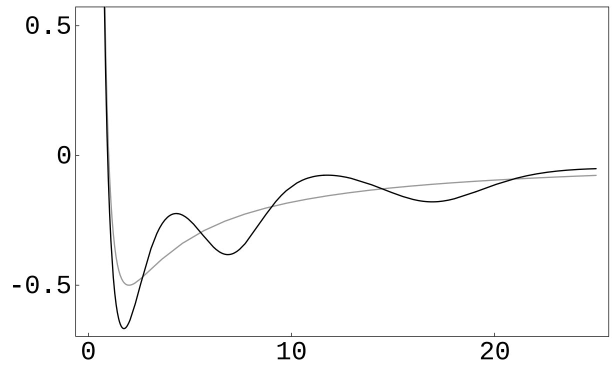

Figure 1: Isospectral SUSY partner potentials $\widetilde{V}(x)$ (black curve) and $V(r) = -2/r$ (gray curve) generated through (18), (20,21) with $n = 4$, $\ell = 1$, $w_0 = -0.1$.

5 Conclusions

Contrasting with the first-order SUSY QM, in which we modify just the ground state energy, the confluent algorithm allows the embedding of levels at any positions on the energy axis. We have shown that the Wronskian formula is still valid in the confluent case. We conclude that the second-order SUSY QM represents a poweful tool in which the right choice of the seed solutions allows us to generate the new potential in a simple way.

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References

[1] J.F. Cariñena, A. Ramos, D.J. Fernández, *Ann. Phys.* **292** (2001) 42

[2] I. Aref'eva, D.J. Fernández, V. Hussin, J. Negro, L.M. Nieto, B.F. Samsonov, *J. Phys. A* **37**, Number 43 (2004)

[3] B. Mielnik, O. Rosas-Ortiz, *J. Phys. A* **37** (2004) 10007

[4] A.A. Andrianov, F. Cannata, *J. Phys. A* **37** (2004) 10297

[5] D.J. Fernández, N. Fernández-García, AIP Conference Proceedings **744** (2005) 236

[6] A.A. Andrianov, M.V. Ioffe, V. Spiridonov, *Phys. Lett. A* **174** (1993) 273: A.A. Andrianov, M.V. Ioffe, F. Cannata, J.P. Dedonder, *Int. J. Mod. Phys. A* **10** (1995) 2683

[7] V.G. Bagrov, B.F. Samsonov, *Phys. Part. Nucl.* **28** (1997) 374; B.F. Samsonov, *Phys. Lett. A* **263** (1999) 274

[8] D.J. Fernández, *Int. J. Mod. Phys. A* **12** (1997) 171; D.J. Fernández, V. Hussin, B. Mielnik, *Phys. Lett. A* **244** (1998) 309; D.J. Fernández, J. Negro, L.M. Nieto, *Phys. Lett. A* **275** (2000) 338

[9] B. Bagchi, A. Ganguly, D. Bhaumik, A. Mitra, *Mod. Phys. Lett. A* **14** (1999) 27

[10] B. Mielnik, L.M. Nieto, O. Rosas-Ortiz, *Phys. Lett. A* **269** (2000) 70

[11] H. Aoyama, M. Sato, T. Tanaka, *Nucl. Phys. B* **619** (2001) 105; M. Plyushchay, *J. Phys. A* **37** (2004) 10375

[12] D.J. Fernández, R. Muñoz, A. Ramos, *Phys. Lett. A* **308** (2003) 11; O. Rosas-Ortiz, R. Muñoz, *J. Phys. A* **36** (2003) 8497

[13] D.J. Fernández, E. Salinas-Hernández, *J. Phys. A* **36** (2003) 2537

[14] D.J. Fernández, *Lett. Math. Phys.* **8** (1984) 337

[15] N.A. Alves, E. Drigo-Filho, *J. Phys. A* **21** (1988) 3215; A. Arai, *Lett. Math. Phys.* **19** (1990) 217

[16] J.O. Rosas-Ortiz, *J. Phys. A* **31** (1998) L507; *ibid* **31** (1998) 10163